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Anderson Localization for One- and Quasi-One-Dimensional Systems

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We prove almost-sure exponential localization of all the eigenfunctions and nondegeneracy of the spectrum for random discrete Schrödinger operators on one- and quasi-one-dimensional lattices. This paper provides a much simpler proof of these results than previous approaches and extends to a much wider class of systems; we remark in particular that the singular continuous spectrum observed in some quasiperiodic systems disappears under arbitrarily small local perturbations of the potential. Our results allow us to prove that, e.g., for strong disorder, the smallest positive Lyapunov exponent of some products of random matrices does not vanish as the size of the matrices increases to infinity.

KEY WORDS:

1. INTRODUCTION

Disordered systems are presently widely studied from the mathematical point of view. One of the challenging questions concerns the Anderson localization theory, which in mathematical terms amounts to studying the nature of the spectrum of random self-adjoint operators, such as for example a discrete Schrödinger equation with a random potential, which is among condensed matter physicists the most popular model for describing the electron propagation in a disordered system. A brief survey of these problems can be found in Ref. 34, whereas Ref. 31 presents a very large bibliography on them. Mathematical reviews will be found in Refs. 6 and 3.

One of the striking predictions of the theory was the prediction by Anderson[(1)] that for sufficiently large disorder all the states should be exponentially localized in any dimension. Localization at large disorder or

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small enough energy for multidimensional systems has recently been proven or announced.[14,9,10,12,33,32]

For one- and quasi-one-dimensional systems, localization was first predicted by Mott and Twose[25] to occur for any energy and arbitrarily small disorder. Such a result for one-dimensional systems was mathematically proven in Refs. 16, 20, 27, 4, 8, 21, 5; it is sufficient to look at these papers to see that the proof of the present paper is much simpler, more general, and more illuminating. For quasi-one-dimensional systems (e.g., the infinite strip), localization was announced in Ref. 15 and also proven in Ref. 23, in a difficult proof, and for such systems our proof is spectacularly simpler and also yields the correct decay rate of eigenfunctions, i.e., the smallest positive Lyapunov exponent.

In this note we first get sufficient conditions ensuring the almost-sure exponential localization for one-dimensional Jacobi matrices as well as discrete Schrödinger equations in a strip. As a result we will obtain that pure-point spectrum, exponential decay of eigenfunctions, and nondegeneracy of the spectrum are almost sure in a large class of one- and quasi-one-dimensional systems. We also show that the Green's function decays at infinity with the smallest Lyapunov exponent as a rate, which was proposed in Ref. 19 in a particular case. Finally, our work allows us to answer a conjecture difficult to attack directly: for classes of random systems at sufficiently strong disorder or low enough energy, the smallest Lyapunov exponent associated to the strip does not vanish as the section of the strip increases to infinity.

The approach of the present paper is basically the following: let us consider for simplicity the one-dimensional case and let us study the equation $H\Psi = E\Psi$. We suppose that the associate Lyapunov exponent is nonzero and thus that all solutions of the equation $H\Psi = E\Psi$ either decay or increase exponentially at $+\infty$ and similarly at $-\infty$. In order to get this property for almost all potential and spectrally almost all E, we use the fact that, under some regularity assumption on the distribution of the potential, the averaged spectral measure of the Hamiltonian is absolutely continuously to the Lebesgue measure.[20,8] From this fact, together with a very clarifying idea of Kotani,[17] implicity contained also in Ref. 5, it follows that for spectrally almost every E the generalized eigenfunctions, which are polynomially bounded on both sides, necessarily decay exponentially on both sides. In fact, it is easy to prove that one can define a Green's function almost everywhere in the spectrum and this in turn implies the nondegeneracy of the spectrum. For the case of the strip the same argument holds and directly yields that the eigenfunctions have an exponential rate of decay equal to or larger than the smallest Lyapunov exponent. This control, together with the property of exponential decay of the eigenfunctions of random Schrödinger operators on $\mathbb{Z}^d$ in the case of strong disorder or large enough energy proven in Ref. 9, shows that, in this case, the smallest positive Lyapunov exponent of the product of random transfer matrices does not vanish as the width of the strip goes to infinity.

A proof of results analogous to ours and taking as we do inspiration from a recent work of Kotani[17] has been developed simultaneously in Refs. 33 and 28. On the other hand one-dimensional results have been also obtained in Ref. 12 in a different way, by using renormalization ideas to extend results from large to arbitrary disorder.

Our approach deals with a much larger class of systems than previous ones, as is emphasized in Section 5 of this paper. In particular it can be applied to the cases of quasiperiodic potentials with arbitrarily small, absolutely continuous local perturbations, leading to pure point spectrum and exponentially decaying eigenfunctions as soon as the smallest Lyapunov exponent is positive; this shows that the known cases of coincidence of positive Lyapunov exponent and singular continuous spectrum are unstable under arbitrarily small local perturbations and that the corresponding singular continuous spectrum is not relevant for physics.

In the following, we first set some definitions and assumptions; then we develop our approach to the exponential localization. In Section 4 we construct the Green's function and show the nondegeneracy of the spectrum. In these first sections we work in a deterministic frame, in the sense that only a finite number of potentials are supposed to be random, the other ones being fixed and assumed to satisfy some properties which are known to be almost sure in most cases of interest. In the last section, dedicated to the description of some applications and extensions, we relate the results of the first sections to the different cases of random systems; we also show the boundedness from below of the smallest Lyapunov exponent when the width of the strip becomes infinite.

2. THE SETTING

Let A be a finite connected subset of $\mathbb{Z}^{d-1}$, where d is a positive integer. The cardinality of A is denoted $|A|$. We consider the lattice $S = A \times \mathbb{Z}$; the one-dimensional case corresponds to $|A| = 1$. We shall use the notation $S_{[a,b]} = A \times [a, b]$. Let $\Omega = \mathbb{R}^S$ be the set of the potentials $V = [V(x)]_{x \in S}$. For $V \in \Omega$ and $\Psi = [\Psi(x)]_{x \in S}$, the discrete Schrödinger operator $H = H^V$ is defined by

$$(H^V \Psi)(x) = \sum_{|y-x|=1} \Psi(y) + V(x) \cdot \Psi(x)$$

where the sum runs on the sites y neighbors of x, and we set for instance free (Dirichlet) boundary conditions. H is a self-adjoint operator on $l^2(S)$ and admits as a core the set of those Ψ with finite support. In the following we denote for simplicity Ψ_n the vector $[\Psi(a, n)]_{a \in A}$ restriction of Ψ to the slice with abscissa n and similarly for V; if Ψ is a solution of the eigenvalue problem $H\Psi = E\Psi$ then Ψ satisfies an equation of the form $(\Psi_{n+1}, \Psi_n) = M_n \cdot (\Psi_n, \Psi_{n-1})$; M_n is a $2|A| \times 2|A|$ matrix called transfer matrix, and is a function of $E - V_n$.

In the following, we fix an interval B of $\mathbb{R}$ and we denote by L the normalized Lebesgue measure on B. Our assumptions for Sections 3 and 4 on the potential are the following.

Hypothesis H1. The potentials outside $S_{[0,1]}$ are given and there exist positive constants α_+ and α_- such that for L-a.e. E there are two $|A|$-dimensional ("contracting") subspaces W_+ and W_- of $\mathbb{R}^{2|A|}$ such that

$$w \in W_+ \Rightarrow \limsup_{n \to +\infty} (1/n) \cdot \log \|M_n \cdot M_{n-1} \cdots M_2 \cdot w\| \leqslant -\alpha_+$$

$$w \in W_- \Rightarrow \limsup_{n \to -\infty} (1/|n|) \cdot \log \|(M_{-1} \cdots M_{n+1} \cdot M_n)^{-1} \cdot w\| \leqslant -\alpha_-$$

Remark 1. Note that the two subspaces W_+ and W_- "live," respectively, on the slices $S_{[1,2]}$ and $S_{[-1,0]}$.

Remark 2. The symplectic nature of the transfer M_n's implies that, under H1,

$$w \notin W_+ \Rightarrow \liminf_{n \to +\infty} (1/n) \cdot \log \|M_n \cdot M_{n-1} \cdots M_2 \cdot w\| \geqslant \alpha_+$$

$$w \notin W_- \Rightarrow \liminf_{n \to -\infty} (1/|n|) \cdot \log \|(M_{-1} \cdots M_{n+1} \cdot M_n)^{-1} \cdot w\| \geqslant \alpha_-$$

Hypothesis H2. The potentials on $S_{[0,1]}$ are random variables with independant distributions which have bounded densities $[\rho_s]_{s \in S_{[0,1]}}$ with respect to the Lebesgue measure.

Remark. In Sections 3 and 4 below, "a.e. V" will thus hold for "almost every realization of $[V(x)]_{x \in S_{[0,1]}}$."

3. THE EXPONENTIAL LOCALIZATION

We are in position to state the first theorem.

Theorem 1. Let H1 and H2 be true. Then H^V has almost surely only pure point spectrum in B and the corresponding eigenvectors decay exponentially with rates at least $\alpha_\pm$ at $\pm\infty$.

Remark. We prove in the next section that the spectrum is non-degenerate.

Proof. Let V be given on S. Let P^V be the resolution of the identity of H^V. For $x \in S$, let $|x\rangle$ denote the vector of the canonical basis of $l^2(S)$ associated to x. Let σ_x be the measure $\langle x| P^V |x\rangle$ (note that σ_x depends on V). Let $\sigma = \sum_{x \in S_{[0,1]}} \sigma_x$. It is well known from spectral theory that for any $\Psi \in l^2(S)$, $\langle \Psi| P^V |\Psi\rangle$ is absolutely continuous with respect to σ. In particular H has pure point spectrum if and only if σ is a point measure. By this remark, Theorem 1 is a mere consequence of the following Proposition 2. ▌

Proposition 2. Let H1 and H2 be true. Let $x \in S_{[0,1]}$ be given. Then, almost-surely, σ_x is a point measure and, for σ_x-a.e. E in $B_\perp$ the corresponding eigenvectors decay exponentially with rates least $\alpha_\pm$ at $\pm\infty$.

Proof. The existence of the subspaces W_+ and W_- considered in H1 does not depend on the value of $V(x)$; thus the existence of such subspaces is known for all E except a set of zero L measure which does not depend on $V(x)$. Thus, for L-a.e. E and any $V(x)$, any solution of $H\Psi = E\Psi$ either increases or decreases exponentially at $+\infty$ with rate at least α_+ and similarly at $-\infty$ with rate α_-.

In order to be in position to achieve the next step, we show that the spectral measure, averaged with respect to the potential, is in fact absolutely continuous with respect to the Lebesgue measure L. This is contained in Proposition 3, the proof of which is given later.

Proposition 3. Let H2 be true. Let $x \in S_{[0,1]}$ be given. Then the measure $\int_{v \in \mathbb{R}} \sigma_x(dE)\, \rho_x(v)\, dv$ is absolutely continuous with respect to $L(dE)$, where v stands for $V(x)$.

This proposition shows that if a set of energies of $V(x)$ is of zero L measure then for a.e. $V(x)$ it has zero σ_x measure. We can now use the idea emphasized by Kotani,[17] namely, that by Proposition 3, a property true for L-a.e. E and independent of $V(x)$ (in our case the exponential behavior of all the solutions of $H\Psi = E\Psi$) is true for a.e. value of $V(x)$ and σ_x-a.e. E in B. Thus, for a.e. $V(x)$ and σ_x-a.e. E, necessarily the polynomially bounded solutions of $H\Psi = E\Psi$ decay exponentially at infinity, with rates at least $\alpha_\pm$. On the other hand, for any value of $V(x)$, it follows from spectral theory that, for σ-a.e. E and thus in particular for σ_x-a.e. E in B, the generalized eigenfunctions of H for the generalized eigenvalue E are polynomially bounded; thus they decay exponentially at infinity, with rates at least $\alpha_\pm$. A fortiori they are in $l^2(S)$ and σ_x is pure point. This ends the proof of Proposition 2. ▌

We are now left with proving Proposition 3, as announced above. It is based on an argument of Ref. 35.

Proof of Proposition 3. Let V be fixed; consider the finite system $S_{[-N,N]}$ $(N>1)$ and denote P_N the resolution of the identity of the restriction H_N of H^V to $S_{[-N,N]}$. Let $\sigma_N = \langle x| \, P_N \, |x\rangle$. For $N \to \infty$, σ_N converges weakly as a measure to σ_x. But σ_N can be written explicitly:

$$\sigma_N(dE) = \sum_k \delta(E - E_k)(\Psi_k(x))^2 \, dE$$

where the sum on k runs on a basis of normed eigenvectors Ψ_k of H_N, the corresponding eigenvalues being denoted E_k. If $\Psi_k(x) = 0$, Ψ remains an eigenvector for the same eigenvalue as $V(x)$ varies and does not contribute to σ_N. We can thus restrict the sum in σ_N to those k such that $\Psi_k(x) \neq 0$ [for all $V(x)$]. In the case when there is a degenerated eigenvalue of degeneracy n, we choose an orthogonal basis of the eigenspace such that at least $n-1$ vectors take the value 0 at site x. Thus we can suppose in the following that in the above expression of $\sigma_N(dE)$ no $\Psi_k(x)$ is 0 and that the eigenvalues E_k are nondegenerate. Each E_k is a monotonous function of $V(x)$ and, as noted in Ref. 8, we have

$$dE_k/dv = [\Psi_k(x)]^2$$

Moreover, as it is easy to see, the two sets of eigenvalues of H obtained, respectively, when $V(x) \to -\infty$ and $V(x) \to +\infty$ are identical except for the eigenvalue corresponding in the limit to the vector $|x\rangle$, which goes to $-\infty$ in one case and to $+\infty$ in the other. This means that, for any given E except a finite set, one and only one among the eigenvalues E_k crosses the value E as $V(x)$ goes from $-\infty$ to $+\infty$. In turn this implies the existence of disjoint open sets 0_k of $\mathbb{R}$ such that the applications $V(x) \to E_k$ are diffeomorphisms from $\mathbb{R}$ to 0_k. This allows to make the following change of variables

$$\begin{aligned}
&\int_{\mathbb{R}} f(E)\, dE \int_{\mathbb{R}} dv \sum_k \delta(E - E_k)[\Psi_k(x)]^2 \\
&\quad = \int_{\mathbb{R}} f(E)\, dE \sum_k \int_{\mathbb{R}} dv\, \delta(E - E_k(v))\, dE_k/dv \\
&\quad = \int_{\mathbb{R}} f(E)\, dE \sum_k \#\, [v \in \mathbb{R}\colon E_k(v) = E] \\
&\quad = \int_{\mathbb{R}} f(E)\, dE
\end{aligned}$$

where f is any continuous function with compact support and v stands for $V(x)$. Thus in view of the expression for $\sigma_N(dE)$, we get

$$\int_{v \in \mathbb{R}} \rho_x(v)\, dv\, \sigma_N(dE) \leqslant \|\rho_x\|_\infty \cdot dE$$

and by the weak convergence of σ_N to σ_x

$$\int_{v \in \mathbb{R}} \rho_x(v)\, dv\, \sigma_x(dE) \leqslant \|\rho_x\|_\infty \cdot dE \quad \blacksquare$$

4. BEHAVIOR OF THE GREEN'S FUNCTION AND NONDEGENERACY OF THE SPECTRUM

In this section we prove the nondegeneracy of the spectrum, under general assumptions. In order to get this result, we first show that the Green's function is defined almost everywhere on the spectrum.

The following proposition is a consequence of the fact that H is a self-adjoint operator on $l^2(S)$:

Proposition 4. Let H1 be true. Then, for any V, the subspaces defined in H1 are such that, for L-a.e. E, $M_1^{-1} \cdot W_+$ and $M_0 \cdot W_-$ are supplementary subspaces of $\mathbb{R}^{2|A|}$, where $M \cdot W$ is the image of the subspace W by the matrix M.

Proof. Let $[V(x)]_{x \in S_{[0,1]}}$ be given. For L-a.e. E the intersection of these subspaces is $\{0\}$, otherwise the solutions of $H\Psi = E\Psi$ such that (Ψ_1, Ψ_0) belongs to $M_1^{-1} \cdot W_+ \cap M_0 \cdot W_-$ would be square integrable (since they decay exponentially) eigenfunctions of H and exist for a set of values of E of positive Lebesgue measure; this is not possible since they would constitute a noncountable set of orthogonal vectors of $l^2(S)$. As W_+ and W_- are $|A|$ dimensional, this proves these subspaces generate $\mathbb{R}^{2|A|}$. $\blacksquare$

Proposition 4 allows us to build the Green's function of H and to prove its exponentially decaying behavior at infinity:

Theorem 5. For L-a.e. V the Green's function $G(x, y; E)$ exists as an unbounded operator admitting as a domain the set of the functions with finite support and it satisfies

$$\forall x, \limsup_{y \to \pm\infty} (1/|y|) \cdot \log |G(x, y; E)| \leqslant -\alpha_\pm$$

Proof. Let E and x be given in the slice of abscissa 0: $x = (a, 0)$ and $|x\rangle = (|a\rangle, |0\rangle)$; we look for a vector $\Phi \in l^2(S)$ such that $(H - E) \cdot \Phi = |x\rangle$;

this amounts to finding an appropriate (Φ_1, Φ_0), since this s sufficient to generate Φ. By H1, we know that a sufficient condition of exponential decay of Φ_n is that

$$(\Phi_1 - |a\rangle, \Phi_0) \in M_0 \cdot W_- \qquad \text{and} \qquad (\Phi_1, \Phi_0) \in (M_1)^{-1} \cdot W_+$$

Proposition 4 tells us that for any V and L-a.e. E, and thus, by Fubini's theorem, for L-a.e. E and for a.e. V, such a (Φ_1, Φ_0) exists and is unique. (Notice that the solution Φ is nonzero even if Φ_0 and Φ_1 are zero). We denote the corresponding exponentially decaying vector Φ by $G(x, \cdot)$. For an arbitrary $x \in S$, $x = (a', n)$ (say, $n > 1$ for simplicity), the same construction can be performed provided one can find (Φ_{n+1}, Φ_n) such that

$$(\Phi_{n+1} - |a'\rangle, \Phi_n) \in M_n \cdots M_1 \cdot M_0 \cdot W_-$$

and

$$(\Phi_{n+1}, \Phi_n) \in M_{n+1} \cdot M_n \cdots M_2 \cdot W_+.$$

As the M_i's are invertible, such a (Φ_{n+1}, Φ_n) still exists and is unique. Thus we get, for L-a.e. E, for a.e. V, the existence of functions $[G(x, \cdot)]_{x \in S}$ such that $(H - E) \cdot G(x, 0) = |x\rangle$. It is then sufficient to write the product $\langle G(x, \cdot)| H - E |G(y, \cdot)\rangle$ to see that $G(x, y) = G(y, x)$. The exponential decay of the functions $G(x, \cdot)$ follows by construction. ▌

We come now to the nondegeneracy of the spectrum.

Theorem 6. Let H1 and H2 be true; then, for a.e. potential, every eigenvalue is nondegenerate.

Proof. Choose $x \in S_{[0,1]}$; fix E and V such that the conclusions of Theorem 5 are valid, and a real number v. Suppose that E is an eigenvalue of $H - v \cdot |x\rangle\langle x|$ and let Ψ be a corresponding normed eigenvector. Let us write the scalar product of $G(y, \cdot)$ with $(H - v \cdot |x\rangle\langle x| - E)\Psi$. As Ψ and $G(y, \cdot)$ decay exponentially at infinity this expression makes sense so that we get

$$\Psi(y) = v \cdot G(x, y) \cdot \Psi(x)$$

which proves that such a Ψ is unique. In other terms, for L-a.e. E, for a.e. V and for any value of v, if E is an eigenvalue of $H - v\,|x\rangle\langle x|$ then the corresponding eigenvector is unique; let us denote this property $\mathbb{P}(E, V + v\,|x\rangle\langle x|)$. If we denote V^* the restriction of the potential to $S \setminus \{x\}$ we get that $\mathbb{P}(E, V)$ is true, for L-a.e. E, for a.e. V^* and independently of the value of $V(x)$. By Fubini's theorem, it is also true for a.e. V^*, for L-a.e. E and for any $V(x)$; by Proposition 3, it remains true for a.e. $V =$

$[V^*, V(x)]$, for σ_x-a.e. E. As x is arbitrary, $\mathbb{P}(E, V)$ is true for a.e. V, for σ-a.e. E. ▌

In the next section we prove (Theorem 8) that, under some assumption of strong disorder, the smallest Lyapunov exponent of the product $\prod M_n$ does not vanish as $|A|$ increases. The following Proposition, which is a consequence of the construction of the Green's function performed in the proof of Theorem 5, relates the smallest Lyapunov exponent to the Green's function.

Proposition 7. Suppose that, in addition to H1, there is a function $\alpha_{\min}(E)$, called smallest Lyapunov exponent, such that for L-a.e. E

$$\exists w_{\pm} \in W_{\pm} \text{ s.t. } \lim_{n \to \pm\infty} (1/|n|) \cdot \log \|M^n \cdot w\| = -\alpha_{\min}(E)$$

$$w \in W_{\pm} \Rightarrow \lim_{n \to \pm\infty} (1/|n|) \cdot \log \|M^n \cdot w\| \leqslant -\alpha_{\min}(E)$$

where M^n stands for $M_n \cdots M_3 \cdot M_2$ if $n > 0$ and for $(M_{-1} \cdot M_{-2} \cdots M_n)^{-1}$ if $n < 0$. Then, for L-a.e. E, for a.e. V, there is a site $x \in S[0, 1]$ such that

$$\lim_{y \to \pm\infty} (1/|y|) \cdot \log |G(x, y)| = -\alpha_{\min}(E)$$

Proof. Consider the family $[G(x, \cdot)]_{x \in S_{[0,1]}}$ constructed in Theorem 5. As $(H - E) \cdot G(x, \cdot) = |x\rangle$ these are $2|A|$ independent vectors. By construction, their restrictions to the positive abscissas lie in the same $|A|$-dimensional subspace, and similarly for their restrictions to the negative abscissas; then they generate these spaces and in particular at least one among them decays at $+\infty$ with rate $\alpha_{\min}(E)$. ▌

5. EXTENSIONS—APPLICATIONS

In the previous sections, the potentials outside the slice $S_{[0,1]}$ were fixed. The results we obtained provide us with tools to deal with many physical situations; the purpose of this section is to describe the applications of our results to random and quasiperiodic potentials.

5.1. Discussion of Hypothesis H1

In many cases of interest, the potentials are supposed to be random in the whole system S, and H1 (which is anyway independent of the potentials on $S_{[0,1]}$) can be shown to be true for almost every realization of the potential outside the slice.

For instance, if the lattice is $\mathbb{Z}$ (i.e., $|A| = 1$), it has been proven[18,30,24] that the Lyapunov exponent of the product $\prod M_n$ is positive (for Lebesgue-a.e. E) as soon as the sequence of ergodic random variables V_n is nondeterministic. Similar properties are expected when A is any finite set; in the case of the strip (i.e., $A \subset \mathbb{Z}$), Lacroix[22] has proven the positivity of the smallest Lyapunov exponent when the $V(x)$'s are independent identically distributed (i.i.d.) random variables, under some weak assumptions on their distribution. In these cases, H1 actually holds for any E, for a.e. potential. If the $V(x)$'s are i.i.d. random variables, and if the smallest Lyapunov exponent is positive, Proposition 4 and the assumptions of Proposition 7 are mere consequences of Oseledec's theorem.

In these cases, H1 is true for a.e. configuration of the potential outside the slice, so that our results (localization, point spectrum, nondegeneracy) remain true for a.e. configuration of V on the whole system.

5.2. Discussion of the Hypothesis H2

Hypothesis H2 was stated in the above form for simplicity. In fact it can easily be weakened and replaced by the following statement:

Hypothesis H2′. The potentials on $S_{[0,1]}$ are random variables with an absolutely continuous joint distribution with respect to the product of the Lebesgue measures $\prod_{s \in S_{[0,1]}} dV(s)$.

The hypothesis H2 is only used in the proof of Proposition 3, where the potentials except $V(x)$ are fixed, so that ρ_x can be replaced everywhere by a conditional expectation $\rho(V(x) \mid [V(s)]_{s \in S_{[0,1]} \setminus \{x\}})$. Moreover, the boundedness of this distribution is not necessary, since, if it is not the case, then for any positive ε and with probability $1 - \varepsilon$, $V(x)$ has a bounded density and all conclusions are valid.

In the cases where the potentials outside $S_{[0,1]}$ are given a probability distribution, the distribution considered in H2′ must be considered as conditioned by the potentials $[V(s)]_{s \notin S_{[0,1]}}$ outside the slice. In particular, in the ergodic case, it is enough to require that this distribution have a part absolutely continuous with respect to the Lebesgue measure; indeed, the above method implies in this case that localization occurs with positive probability, and it is known by ergodicity that it can occur only with probability 0 or 1 on the potential V.[20]

5.3. Lower Bound on the Smallest Lyapunov Exponent

For sufficiently strong disorder or at low enough energy, it is expected that the Green's function exhibits (Lebesgue-almost everywhere on the spectrum) an exponential decay at infinity with a rate uniformly bounded below as $|A|$ increases. In some particular cases, this follows, as shown in Ref. 9, from the results of Ref. 13. This bound on the decay rate, together with Proposition 7, gives information on the behavior of the smallest Lyapunov exponent $\alpha_{\min}$ as the section A increases to $\mathbb{Z}^{d-1}$:

Theorem 8. Suppose V is sufficiently random or E low enough in the sense described above. Then the smallest Lyapunov exponent of the transfer matrices product $M_n \cdot M_{n-1} \cdots M_1$ remains positively bounded below as $A \nearrow \mathbb{Z}^{d-1}$.

Proof. Let E be given in the spectrum. By Proposition 7, we know that, with probability 1, some $G(x, \cdot)$ decays at $+\infty$ with rate $\alpha_{\min}(E)$. By Ref. 9 we know that, with full probability, it is possible for any x to construct a function G_x such that $(H-E) \cdot G_x = |x\rangle$ and to prove that it decays at infinity with an exponential rate bounded below by a positive constant independent of the geometry of $|A|$. Almost surely, E is not an eigenvalue for H, so that $G(x, \cdot)$ and G_x coincide; thus $\alpha_{\min}(E)$ is bounded below by a positive constant. ∎

5.4. Perturbations of Deterministic Systems

It is known that some deterministic systems exhibit positive Lyapunov exponents, which means that they satisfy H1, but give rise to singular continuous spectrum. One of the known examples is the one-dimensional system where the potential is given by $V(n) = \lambda \cdot \cos[2\pi(\alpha n + \theta)]$, $\lambda > 2$ and α Liouville.[2] A consequence of our results is that, under an arbitrarily small continuous perturbation of the potentials on the slice, the singular continuous spectrum is almost surely replaced by a pure point spectrum with eigenfunctions decaying at least with the Lyapunov exponent as a rate!

5.5. Other Possible Asymptotic Behaviors

The exponential nature of the decay assumed in H1 (related to the exponential increase of the norm of the matrices product) is not essential; neither is it universal, and some models give rise to different asymptotic behaviors. For instance, a faster than exponential decay of the matrices product is believed to occur for self-similar potentials, with a behavior $\|M_n \cdot M_{n-1} \cdots M_1\| \sim n^{c \cdot n}$; this behavior is related to the discretization of our model and, in continuous models, behaviors (e.g., of the Green's function) in $e^{n^{\xi}}$, $\xi > 1$, are expected. On the contrary, if the random potential decays at infinity like $1/|n|^{\alpha}$ the matrices product is found to increase like $\exp(c \cdot |n|^{1-2\alpha})$ for $0 < \alpha < 1/2$.[29,11]

In these two cases, if the analogs of H1 could be recovered one would find the eigenstates to decay like $n^{-c \cdot n}$ [resp. $\exp(-c \cdot |n|^{1-2\alpha})$]; this latter behavior has been proven previously in Ref. 29.

In Section 3 the polynomial bound on the generalized eigenstates allowed us to deduce that, as they exponentially either increase or decay, they necessarily decay. Actually, one knows that the generalized eigenfunctions of H are indeed bounded by $C \cdot |n|^{1/2+\varepsilon}$ (for any $\varepsilon > 0$). Thus the exponential decay of $\|M_n \cdot M_{n-1} \cdots M_2 \cdot w\|$ in H1 can be replaced by any decay faster than $|n|^{-(1/2+\varepsilon)}$, leading to the same asymptotic behavior for the eigenfunctions (and in particular to pure point spectrum). The same behavior holds for the Green's functions; the nondegeneracy of the spectrum follows also: the behaviors of the Green's functions and of the eigenfunctions are sufficient to ensure the convergence of the scalar product $\langle G(y, \cdot) \mid (H - v \cdot |x\rangle\langle x| - E) \mid \Psi\rangle$ in the proof of Theorem 6.

A more detailed analysis allows us to consider the cases when the decay of $\|M_n \cdot M_{n-1} \cdots M_2 \cdot w\|$ in H1 is weaker than $|n|^{-1/2}$, for instance like $|n|^{\varepsilon}$, $0 < \varepsilon < 1/2$; in these cases one guesses continuous spectrum, but we still apply our method to describe the behavior of the generalized eigenfunctions. Indeed the spectral theorem yields that the generalized eigenfunctions cannot increase faster than $|n|^{\delta}$ for $\delta > 0$ (in the sense that $\Psi_n^2 + \Psi_{n+1}^2 > C \cdot |n|^{\delta}$). Now, as in the proof of Theorem 1, using Proposition 3 and the Wronskian property we get that, for a.e. potential, for σ-a.e. value of E the corresponding generalized eigenfunctions necessarily increase like $|n|^{\varepsilon}$ or decrease like $|n|^{-\varepsilon}$ at infinity, so that for a.e. potential, σ-a.e. E, they decay like $|n|^{-\varepsilon}$; thus the spectrum is continuous. On the other hand, by Fubini's theorem, for Lebesgue almost every E the solutions of $H\Psi = E\Psi$ behave like $|n|^{\varepsilon}$ at $+\infty$, $-\infty$ or both; consequently the measure σ is singular with respect to the Lebesgue measure.[26] This provides a case where a singular continuous spectral measure occurs with probability 1, although distributions of the potentials are continuous. A behavior in $|n|^{\varepsilon}$ of the matrices product is proven in a weaker sense if the random potential V_n decays at infinity like $|n|^{-1/2}$;[11] the singular continuous spectrum has been proven for this case[7] by other means since the weak version of H1 available in this case does not provide the splitting necessary to predict the behavior of the generalized eigenstates.

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